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INVARIANTS OF HYPERBOLIC EQUATIONS: SOLUTION OF THE LAPLACE PROBLEM

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This paper gives a solution of the Laplace problem, which consists of finding all invariants of the hyperbolic equations and constructing a basis of the invariants. Three new invariants of the first and second orders are found, and invariant-differentiation operators are constructed. It is shown that the new invariants, together with the two invariants detected by Ovsyannikov, form a basis such that any invariant of any order is a function of the basis invariants and their invariant derivatives.

Key words: *Laplace invariants, integration of hyperbolic equations, equivalence transformations, semi-invariants.*

Introduction. The famous Laplace invariants h and k appeared for the first time in Laplace's paper (1773) on the theory of integration of linear hyperbolic differential equations with two independent variables. It is more proper, however, to call the quantities h and k *semi-invariant* because they are invariant only under a linear substitution of a dependent variable.

In his fundamental paper [1] on the integration of linear hyperbolic second-order partial differential equations

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$

Laplace considered the following quantities:

$$h = a_x + ab - c, \qquad k = b_y + ab - c.$$

The quantities h and k do not change under a linear transformation of the dependent variable $u = \varphi(x,y)v$ and, hence, were later called *Laplace invariants.* The term *semi-invariant* is more accurate [2] since h and k are invariant only under a subgroup of the equivalence transformation group rather than the entire group. The term *semi-invariant* for quantities that are invariant under subgroups was proposed by Laguerre [3] according to the general theory of Cayley's invariants. The question of the presence or absence of other invariants remained opened.

Nearly 200 years had passed before Ovsyannikov [4], studying the problem of group classification of hyperbolic equations, found two true invariants

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x\,\partial y},$$

which do not change under all equivalence transformations. At that time, the general approach to constructing invariants of systems of equations with an infinite equivalence transformation group had not been developed, and, hence, the problem of whether all invariants are exhausted by the quantities found remained open. Thus the following problem arose (I call it the *Laplace problem*): to find all invariants of hyperbolic equations and to construct a basis of the invariants.

A general method for constructing invariants of systems of linear and nonlinear equations using infinite equivalence transformation groups was recently developed in [2] (see also [5]). This method was then applied to several linear and nonlinear equations. In particular, applied to the parabolic equation

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0$$

this method yielded the following invariant [6]:

$$K = b^2 a_x/2 + (a_t + a a_{xx} - a_x^2)b + (a a_x - ab)b_x - a b_t - a^2 b_{xx} + 2a^2 c_x.$$

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In the present paper, we solve the Laplace problem of the invariants of hyperbolic equations. To construct a basis of the invariants, one first computes all invariants up to the second order, inclusive, and then finds the next three new invariants:

$$I = \frac{p_x p_y}{h}, \qquad N = \frac{1}{p_x} \frac{\partial}{\partial x} \ln\left|\frac{p_x}{h}\right|, \qquad H = \frac{1}{p_y} \frac{\partial}{\partial y} \ln\left|\frac{p_y}{h}\right|.$$

After that, the general invariant-differentiation operator

$$\mathcal{D} = F(p, I)\, \frac{1}{p_x}\, D_x + G(p, I)\, \frac{1}{p_y}\, D_y$$

is computed and it is proved that of the new invariants and Ovsyannikov invariants, it is possible construct a basis of all invariants so that any invariant of any order is a function of the basis invariants and their invariant derivatives.

A detailed description of the method used below can be found in [2], and in [5, Sec. 10], where the method is illustrated on examples of calculations of invariants of algebraic and ordinary differential equations.

1. Semi-Invariants. We consider hyperbolic equations with two independent variables x and y

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \tag{1}$$

where the subscripts denote partial derivatives ($u_x = \partial u/\partial x$, etc.).

An equivalence transformation of Eqs. (1) is a reversible transformation of the variables

$$\bar{x} = f(x, y, u), \qquad \bar{y} = g(x, y, u), \qquad \bar{u} = h(x, y, u)$$

under which Eq. (1) with any coefficients a, b, and c remains linear and homogeneous. In this case, generally speaking, the transformed equation has new coefficients $\bar{a}$, $\bar{b}$, and $\bar{c}$.

The set of all equivalence transformations of Eqs. (1) is a infinite group consisting of a linear transformation of the dependent variable

$$u = \varphi(x, y)v, \qquad \varphi(x, y) \neq 0 \tag{2}$$

and the following reversible substitutions of the independent variables:

$$\bar{x} = f(x), \qquad \bar{y} = g(y), \tag{3}$$

where $f(x)$, $g(y)$, and $\varphi(x, y)$ are arbitrary functions and $v(\bar{x}, \bar{y})$ is a new dependent variable. Two equations of the form of (1) are called *equivalent* if they can be related to each other by an appropriate equivalence transformation (2), (3).

Functions $J = J(a, b, c, a_x, a_y, \ldots)$ of the variables a, b, and c and their derivatives will be called *semi-invariants* of Eq. (1) if these functions are invariant only under transformation (2). In this section, we find all semi-invariants. The apparent semi-invariants x and y are not considered here.

We set $\varphi(x, y) \approx 1 + \varepsilon\sigma(x, y)$, where ε is a small parameter and consider the infinitesimal transformation (2)

$$u \approx [1 + \varepsilon\sigma(x, y)]v.$$

Then, the corresponding infinitesimal transformation of the derivatives is given by

$$u_x \approx (1 + \varepsilon\sigma)v_x + \varepsilon\sigma_x v, \qquad u_y \approx (1 + \varepsilon\sigma)v_y + \varepsilon\sigma_y v, \qquad u_{xy} \approx (1 + \varepsilon\sigma)v_{xy} + \varepsilon\sigma_y v_x + \varepsilon\sigma_x v_y + \varepsilon\sigma_{xy} v.$$

Therefore, we have

$$u_{xy} + au_x + bu_y + cu \approx (1 + \varepsilon\sigma)v_{xy} + \varepsilon\sigma_y v_x + \varepsilon\sigma_x v_y + \varepsilon\sigma_{xy} v$$

$$+ (1 + \varepsilon\sigma)av_x + \varepsilon\sigma_x av + (1 + \varepsilon\sigma)bv_y + \varepsilon\sigma_y bv(1 + \varepsilon\sigma)cv$$

and arrive at the infinitesimal transformation of Eq. (1):

$$v_{xy} + (a + \varepsilon\sigma_y)v_x + (b + \varepsilon\sigma_x)v_y + [c + \varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y)]v = 0.$$

From this it is obvious that the coefficients of Eq. (1) are subjected to the infinitesimal transformation

$$\bar{a} \approx a + \varepsilon\sigma_y, \qquad \bar{b} \approx b + \varepsilon\sigma_x, \qquad \bar{c} \approx c + \varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y)$$

and yield the following infinitesimal operator:

$$Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + (\sigma_{xy} + a\sigma_x + b\sigma_y)\frac{\partial}{\partial c}. \tag{4}$$

Let us first consider the problem of the existence of semi-invariants of the form $J = J(a, b, c)$. The invariance criterion of $Z(J) = 0$ under transformation (2) is written as

$$\sigma_y \frac{\partial J}{\partial a} + \sigma_x \frac{\partial J}{\partial b} + (\sigma_{xy} + a\sigma_x + b\sigma_y)\frac{\partial J}{\partial c} = 0.$$

Since the function $\sigma(x, y)$ is arbitrary, by equating the coefficients at σ_{xy}, σ_x, and σ_y to zero, this equation is split into the following three equations:

$$\frac{\partial J}{\partial c} = 0, \qquad \frac{\partial J}{\partial b} = 0, \qquad \frac{\partial J}{\partial a} = 0.$$

Hence, $J = \text{const}$, so that there are no invariants of the form of $J = J(a, b, c)$ other than the apparent constant J.

Therefore, it is necessary to consider first-order differential invariants, i.e., functions of the form $J(a, b, c, a_x, a_y, b_x, b_y, c_x, c_y)$. To find such invariants, one needs to use the extended operator (4)

$$Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + (\sigma_{xy} + a\sigma_x + b\sigma_y)\frac{\partial}{\partial c} + \sigma_{xy} \frac{\partial}{\partial a_x} + \sigma_{yy} \frac{\partial}{\partial a_y} + \sigma_{xx} \frac{\partial}{\partial b_x} + \sigma_{xy} \frac{\partial}{\partial b_y}$$

$$+ (\sigma_{xxy} + a\sigma_{xx} + a_x\sigma_x + b\sigma_{xy} + b_x\sigma_y)\frac{\partial}{\partial c_x} + (\sigma_{xyy} + a\sigma_{xy} + a_y\sigma_x + b\sigma_{yy} + b_y\sigma_y)\frac{\partial}{\partial c_y}$$

and solve the equation

$$ZJ(a, b, c, a_x, a_y, b_x, b_y, c_x, c_y) = 0.$$

In this equation, the vanishing of the first coefficients at σ_{xxy} and σ_{xyy} and then at σ_{xx} and σ_{yy} yields the following four equations:

$$\frac{\partial J}{\partial c_x} = 0, \qquad \frac{\partial J}{\partial c_y} = 0, \qquad \frac{\partial J}{\partial b_x} = 0, \qquad \frac{\partial J}{\partial a_y} = 0,$$

whence it follows that $J = J(a, b, c, a_x, b_y)$. Now, the vanishing of the coefficients at σ_{xy}, σ_x, σ_y yields the following system of three equations:

$$\frac{\partial J}{\partial c} + \frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} = 0, \qquad \frac{\partial J}{\partial b} + a\,\frac{\partial J}{\partial c} = 0, \qquad \frac{\partial J}{\partial a} + b\,\frac{\partial J}{\partial c} = 0.$$

The last two equations of this system yield $J = J(\lambda, a_x, b_y)$, where $\lambda = ab - c$. Then, the first equation becomes

$$\frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} - \frac{\partial J}{\partial \lambda} = 0.$$

This equation has two independent solutions:

$$J_1 = a_x - b_y, \qquad J_2 = a_x + \lambda \equiv a_x + ab - c.$$

Using the notation $h = J_2$ and $k = J_2 - J_1$, we obtain two functionally independent *semi-invariants*, namely *Laplace invariants:*

$$h = a_x + ab - c, \qquad k = b_y + ab - c. \tag{5}$$

Lemma 1. *Semi-invariants depending on the higher derivatives of a, b, and c are functions of the Laplace invariants* (5) *and their derivatives of any order with respect to x and y.*

Lemma 1 is proved by comparing the number of invariants obtained by differentiation of the Laplace invariants (5) with the difference between the number of derivatives of the corresponding order of the functions a, b, and c and the number of derivatives of the same order of the function σ.

2. Second-Order Invariants. A function $J = J(x, y, a, b, c, a_x, a_y, \ldots)$ of the variables x and y, the coefficients a, b, and c, and their derivatives of any order is called *an invariant* of Eq. (1) if it does not change under the general equivalence transformation (2), (3) of the dependent and independent variables. According to Lemma 1 proved above, to determine the most general invariant, it suffices to use functions of the form

$$J(x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, h_{xy}, h_{yy}, k_{xx}, k_{xy}, k_{yy}, \ldots) \tag{6}$$

and to subject them to the condition of invariance under transformations (3) of independent variables.

The infinitesimal transformation (3) of the variable x has the form

$$\bar{x} \approx x + \varepsilon\xi(x) \tag{7}$$

and gives

$$u_x \approx (1 + \varepsilon\xi')u_{\bar{x}}, \qquad u_y = u_{\bar{y}}, \qquad u_{xy} \approx (1 + \varepsilon\xi')u_{\bar{x}\bar{y}},$$

where $\xi' = d\xi(x)/dx$. These formulas leads to the infinitesimal transformation of Eq. (1)

$$(1 + \varepsilon\xi')u_{\bar{x}\bar{y}} + a(1 + \varepsilon\xi')u_{\bar{x}} + bu_{\bar{y}} + cu = 0,$$

which one can be written, with accuracy up to the first order in ε, in the form of (1):

$$u_{\bar{x}\bar{y}} + au_{\bar{x}} + (b - \varepsilon\xi' b)u_{\bar{y}} + (c - \varepsilon\xi' c)u = 0.$$

This leads to the following infinitesimal transformations of the coefficients of Eq. (1):

$$\bar{a} \approx a, \qquad \bar{b} \approx b - \varepsilon\xi' b, \qquad \bar{c} \approx c - \varepsilon\xi' c. \tag{8}$$

The infinitesimal transformations (7) and (8) define the operator

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi' b\frac{\partial}{\partial b} + \xi' c\frac{\partial}{\partial c}. \tag{9}$$

The extension of the operator (9) to a_x and b_y has the form

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi'(x)\Big[b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} + a_x\frac{\partial}{\partial a_x} + b_y\frac{\partial}{\partial b_y}\Big]$$

and specifies the following action on the Laplace invariants:

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi'(x)\Big[h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k}\Big]. \tag{10}$$

Here we wish to find all invariants (6) that depend on the derivatives of h and k up to the second order, inclusive. Therefore, we calculate the second extension of the operator (10) using the general procedure and obtain

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi' h\frac{\partial}{\partial h} + \xi' k\frac{\partial}{\partial k} + (\xi'' h + 2\xi' h_x)\frac{\partial}{\partial h_x} + (\xi'' k + 2\xi' k_x)\frac{\partial}{\partial k_x}$$

$$+ \xi' h_y\frac{\partial}{\partial h_y} + \xi' k_y\frac{\partial}{\partial k_y} + (\xi''' h + 3\xi'' h_x + 3\xi' h_{xx})\frac{\partial}{\partial h_{xx}} + (\xi'' h_y + 2\xi' h_{xy})\frac{\partial}{\partial h_{xy}}$$

$$+ \xi' h_{yy}\frac{\partial}{\partial h_{yy}} + (\xi''' k + 3\xi'' k_x + 3\xi' k_{xx})\frac{\partial}{\partial k_{xx}} + (\xi'' k_y + 2\xi' k_{xy})\frac{\partial}{\partial k_{xy}} + \xi' k_{yy}\frac{\partial}{\partial k_{yy}}.$$

Now, as in Sec. 1, we use the infinite-dimensional nature of the examined Lie algebra of operators, namely the fact that the function $\xi(x)$ and all its derivatives $\xi'(x)$, $\xi''(x)$, and $\xi'''(x)$ are arbitrary. Therefore, the above extended operator splits into the following four operators, obtained by separating the coefficients at different derivatives of the function $\xi(x)$:

$$X_\xi = \frac{\partial}{\partial x}, \qquad X_{\xi'''} = h\frac{\partial}{\partial h_{xx}} + k\frac{\partial}{\partial k_{xx}},$$

$$\begin{aligned} X_{\xi'} &= h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k} + 2h_x\frac{\partial}{\partial h_x} + h_y\frac{\partial}{\partial h_y} + 2k_x\frac{\partial}{\partial k_x} + k_y\frac{\partial}{\partial k_y} + 3h_{xx}\frac{\partial}{\partial h_{xx}} \\ &\quad + 2h_{xy}\frac{\partial}{\partial h_{xy}} + h_{yy}\frac{\partial}{\partial h_{yy}} + 3k_{xx}\frac{\partial}{\partial k_{xx}} + 2k_{xy}\frac{\partial}{\partial k_{xy}} + k_{yy}\frac{\partial}{\partial k_{yy}}, \end{aligned} \tag{11}$$

$$X_{\xi''} = h\frac{\partial}{\partial h_x} + k\frac{\partial}{\partial k_x} + 3h_x\frac{\partial}{\partial h_{xx}} + h_y\frac{\partial}{\partial h_{xy}} + 3k_x\frac{\partial}{\partial k_{xx}} + k_y\frac{\partial}{\partial k_{xy}}.$$

Similarly, the infinitesimal transformation (3) of the variable y

$$\bar{y} \approx y + \varepsilon\eta(y)$$

leads to the operator

$$Y = -\eta(y)\frac{\partial}{\partial y} + \eta'(y)\Big[h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k}\Big], \tag{12}$$

whose second extension splits into the following operators:

$$Y_\eta = \frac{\partial}{\partial y}, \qquad Y_{\eta'''} = h\,\frac{\partial}{\partial h_{yy}} + k\,\frac{\partial}{\partial k_{yy}},$$

$$\begin{aligned} Y_{\eta'} = h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k} + h_x\,\frac{\partial}{\partial h_x} + 2h_y\,\frac{\partial}{\partial h_y} + k_x\,\frac{\partial}{\partial k_x} + 2k_y\,\frac{\partial}{\partial k_y} + h_{xx}\,\frac{\partial}{\partial h_{xx}} \\ + 2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 3h_{yy}\,\frac{\partial}{\partial h_{yy}} + k_{xx}\,\frac{\partial}{\partial k_{xx}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + 3k_{yy}\,\frac{\partial}{\partial k_{yy}}, \\ Y_{\eta''} = h\,\frac{\partial}{\partial h_y} + k\,\frac{\partial}{\partial k_y} + h_x\,\frac{\partial}{\partial h_{xy}} + 3h_y\,\frac{\partial}{\partial h_{yy}} + k_x\,\frac{\partial}{\partial k_{xy}} + 3k_y\,\frac{\partial}{\partial k_{yy}}. \end{aligned} \qquad (13)$$

From the conditions of invariance under translations along x and y [$X_\xi(J) = 0$ and $Y_\eta(J) = 0$], it follows that J in (6) does not depend on x and y. In addition, from the form of the operators (13) it can be seen that both equations

$$h = 0, \qquad k = 0 \qquad (14)$$

are invariant under (13). Below, we assume that h and k do not vanish simultaneously, for example, $h \neq 0$. The equation $X_{\xi'}J = 0$ for the function $J(h,k)$ gives one of the Ovsyannikov invariants

$$p = k/h. \qquad (15)$$

It is easy to verify that the quantity p satisfies the invariance criterion for all operators (11) and (13). Next, from the equations $X_{\xi'''}(J) = 0$ and $Y_{\eta'''}(J) = 0$, it follows that h_{xx}, h_{yy}, k_{xx}, and k_{yy} can appear in the second-order invariants (6) only in the following combinations:

$$r = k_{xx} - p\,h_{xx}, \qquad s = k_{yy} - p\,h_{yy}.$$

Thus, the general form of the second-order invariants (6) reduces to the dependence

$$J(h, p, h_x, h_y, k_x, k_y, h_{xy}, k_{xy}, r, s). \qquad (16)$$

The invariance conditions for functions of the form of (16) are given by

$$X_{\xi'}(J) = 0, \qquad X_{\xi''}(J) = 0, \qquad Y_{\eta'}(J) = 0, \qquad Y_{\eta''}(J) = 0, \qquad (17)$$

where the operators $X_{\xi'}$, $X_{\xi''}$, $Y_{\eta'}$, and $Y_{\eta''}$ are written in the variables included in (16), namely:

$$\begin{aligned} X_{\xi'} = h\,\frac{\partial}{\partial h} + 2h_x\,\frac{\partial}{\partial h_x} + h_y\,\frac{\partial}{\partial h_y} + 2k_x\,\frac{\partial}{\partial k_x} + k_y\,\frac{\partial}{\partial k_y} + 2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + 3r\,\frac{\partial}{\partial r} + s\,\frac{\partial}{\partial s}, \\ X_{\xi''} = h\,\frac{\partial}{\partial h_x} + ph\,\frac{\partial}{\partial k_x} + h_y\,\frac{\partial}{\partial h_{xy}} + k_y\,\frac{\partial}{\partial k_{xy}} + 3(k_x - ph_x)\,\frac{\partial}{\partial r}, \\ Y_{\eta'} = h\,\frac{\partial}{\partial h} + h_x\,\frac{\partial}{\partial h_x} + 2h_y\,\frac{\partial}{\partial h_y} + k_x\,\frac{\partial}{\partial k_x} + 2k_y\,\frac{\partial}{\partial k_y} + 2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + r\,\frac{\partial}{\partial r} + 3s\,\frac{\partial}{\partial s}, \\ Y_{\eta''} = h\,\frac{\partial}{\partial h_y} + ph\,\frac{\partial}{\partial k_y} + h_x\,\frac{\partial}{\partial h_{xy}} + k_x\,\frac{\partial}{\partial k_{xy}} + 3(k_y - ph_y)\,\frac{\partial}{\partial s}. \end{aligned} \qquad (18)$$

The operators (18) satisfy the commutation relations

$$[X_{\xi'}, X_{\xi''}] = -X_{\xi''}, \qquad [X_{\xi'}, Y_{\eta'}] = 0, \qquad [X_{\xi'}, Y_{\eta''}] = 0,$$

$$[X_{\xi''}, Y_{\eta'}] = 0, \qquad [X_{\xi''}, Y_{\eta''}] = 0, \qquad [Y_{\eta'}, Y_{\eta''}] = -Y_{\eta''}$$

and, hence, form a basis of the Lie algebra. The commutators given above show that it is convenient to solve system (17) beginning with the following equations (see [5, Secs. 4.5.3]):

$$\begin{aligned} X_{\xi''}(J) = h\,\frac{\partial J}{\partial h_x} + ph\,\frac{\partial J}{\partial k_x} + h_y\,\frac{\partial J}{\partial h_{xy}} + k_y\,\frac{\partial J}{\partial k_{xy}} + 3(k_x - ph_x)\,\frac{\partial J}{\partial r} = 0, \\ Y_{\eta''}(J) = h\,\frac{\partial J}{\partial h_y} + ph\,\frac{\partial J}{\partial k_y} + h_x\,\frac{\partial J}{\partial h_{xy}} + k_x\,\frac{\partial J}{\partial k_{xy}} + 3(k_y - ph_y)\,\frac{\partial J}{\partial s} = 0. \end{aligned} \qquad (19)$$

Integration of the characteristic system

$$\frac{dh_x}{h} = \frac{dk_x}{ph} = \frac{dh_{xy}}{h_y} = \frac{dk_{xy}}{k_y} = \frac{dr}{3(k_x - ph_x)}$$

for the first equation (19) shows that the quantities h, p, h_y, k_y, and s can appear in J only in the following combinations:

$$\lambda = k_x - ph_x, \qquad \tau = hh_{xy} - h_x h_y, \qquad \nu = phk_{xy} - k_x k_y, \qquad \omega = hr - 3\lambda h_x.$$

Thus, the second equation of (19) reduces to the equation

$$Y_{\eta''}(J) = h\,\frac{\partial J}{\partial h_y} + ph\,\frac{\partial J}{\partial k_y} + 3(k_y - ph_y)\,\frac{\partial J}{\partial s} = 0,$$

whose integration shows that the general solution of Eqs. (19) has the form $J = J(h, p, \lambda, \mu, \tau, \nu, \omega, \rho)$, where

$$\begin{gathered} \lambda = k_x - ph_x, \qquad \mu = k_y - ph_y, \qquad \tau = hh_{xy} - h_x h_y, \\ \nu = phk_{xy} - k_x k_y, \qquad \omega = hr - 3\lambda h_x, \qquad \rho = hs - 3\mu h_y. \end{gathered} \tag{20}$$

Solving the equation $(X_{\xi'} - Y_{\eta'})(J) = 0$ written in the variables h, p, λ, μ, τ, ν, ω, and ρ

$$(X_{\xi'} - Y_{\eta'})(J) = \lambda\,\frac{\partial J}{\partial \lambda} - \mu\,\frac{\partial J}{\partial \mu} + 2\omega\,\frac{\partial J}{\partial \omega} - 2\rho\,\frac{\partial J}{\partial \rho} = 0,$$

we have $J = J(h, p, m, \tau, \nu, n, N)$, where

$$m = \lambda\mu, \qquad n = \omega\rho, \qquad N = \omega/\lambda^2. \tag{21}$$

To complete the integration of system (17), it is necessary to solve the equation

$$X_{\xi'}(J) = h\,\frac{\partial J}{\partial h} + 3\tau\,\frac{\partial J}{\partial \tau} + 3\nu\,\frac{\partial J}{\partial \nu} + 3m\,\frac{\partial J}{\partial m} + 6n\,\frac{\partial J}{\partial n} = 0.$$

As a result, we obtain the following six independent invariants of the second order:

$$p = \frac{k}{h}, \quad q = \frac{\tau}{h^3}, \quad Q = \frac{\nu}{h^3}, \quad N = \frac{\omega}{\lambda^2}, \quad M = \frac{n}{h^6}, \quad I = \frac{m}{h^3} \tag{22}$$

provided that $h \neq 0$ and $\lambda \neq 0$. We note that each of the equations

$$\lambda \equiv k_x - ph_x = 0, \qquad \mu \equiv k_y - ph_y = 0 \tag{23}$$

is invariant. In our calculations, we omit the cases where Eqs. (23) and (14) are satisfied.

Let us now write invariants (22) in terms of the Laplace semi-invariants h and k and the Ovsyannikov invariant $p = k/h$. From the equations

$$k_x - ph_x \equiv (hk_x - kh_x)/h = hp_x, \qquad k_y - ph_y \equiv (hk_y - kh_y)/h = hp_y$$

we have

$$\begin{gathered} \lambda = k_x - ph_x = hp_x, \qquad \mu = k_y - ph_y = hp_y, \\ r = k_{xx} - ph_{xx} = hp_{xx} + 2h_x p_x, \qquad \omega = h^2 p_{xx} - hh_x p_x, \\ s = k_{yy} - ph_{yy} = hp_{yy} + 2h_y p_y, \qquad \rho = h^2 p_{yy} - hh_y p_y. \end{gathered} \tag{24}$$

From this it is easy to see that

$$q = \frac{\tau}{h^3} = \frac{h_{xy}}{h^2} - \frac{h_x h_y}{h^3} \equiv \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x\,\partial y} \tag{25}$$

and $Q = p^3\tilde{q}$. Here $\tilde{q}$ is an invariant (since p^3 is an invariant) and has the form

$$\tilde{q} = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x\,\partial y}. \tag{26}$$

Next, we note that the invariant

$$M = \frac{\omega}{h^6} = \Big(\frac{p_x}{h}\Big)_x \Big(\frac{p_y}{h}\Big)_y$$

can be replaced by another invariant, namely,

$$H = \rho/\mu^2 \tag{27}$$

by using equalities (21)–(24) and the identity $M = NHI^2$, which follows from the relation

$$NH = \frac{\omega\rho}{\lambda^2\mu^2} = \frac{\omega\rho}{h^4 p_x^2 p_y^2} = \frac{\omega\rho}{h^6 I^2}.$$

Taking into account the definitions (20)–(22) and using Eqs. (24), we obtain

$$N = \frac{\omega}{\lambda^2} = \frac{h(k_{xx} - ph_{xx})}{(k_x - ph_x)^2} - \frac{3h_x}{k_x - ph_x} = \frac{p_{xx}}{p_x^2} - \frac{h_x}{hp_x} = \frac{1}{p_x}\Big(\ln\Big|\frac{p_x}{h}\Big|\Big)_x. \tag{28}$$

Similarly, we rewrite invariant (27) as

$$H = \frac{\rho}{\mu^2} = \frac{p_{yy}}{p_y^2} - \frac{h_y}{hp_y} = \frac{1}{p_y}\Big(\ln\Big|\frac{p_y}{h}\Big|\Big)_y. \tag{29}$$

As a result, we have

$$I = \frac{\lambda\mu}{h^3} = \frac{p_x p_y}{h}\,. \tag{30}$$

Collecting together invariants (15), (25), (26), and (28)–(30), we finally arrive at the following complete set of second-order invariants for Eq. (1):

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x\,\partial y}, \qquad \tilde{q} = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x\,\partial y}; \tag{31}$$

$$N = \frac{1}{p_x}\,\frac{\partial}{\partial x}\ln\Big|\frac{p_x}{h}\Big|, \qquad H = \frac{1}{p_y}\,\frac{\partial}{\partial y}\ln\Big|\frac{p_y}{h}\Big|, \qquad I = \frac{p_x p_y}{h}. \tag{32}$$

In addition, there are the following individually invariant equations (14) and (23):

$$h = 0, \qquad k = 0, \qquad k_x - ph_x = 0, \qquad k_y - ph_y = 0.$$

3. Invariant Differentiation. We now find invariant differentiations that transform each invariant of Eq. (1) into the invariants of the same equation. Recall that for any group specified by means of the infinitesimal operators

$$X_\nu = \xi_\nu^i(x,u)\,\frac{\partial}{\partial x^i} + \eta_\nu^\alpha(x,u)\,\frac{\partial}{\partial u^\alpha}$$

with n independent variables $x = (x^1, \ldots, x^n)$, there exist n invariant differentiations of the form (see [7, Sec. 7] and also [5, Secs. 8.3.5])

$$\mathcal{D} = f^i D_i. \tag{33}$$

Their coefficients have the form $f^i = f^i(x, u, u_{(1)}, u_{(2)}, \ldots)$ and are found by solving the differential equations

$$X_\nu(f^i) = f^j D_j(\xi_\nu^i), \qquad i = 1, \ldots, n. \tag{34}$$

In our case, the operators X_ν are (10) and (12). The invariant differentiation operator (33) can be written as

$$\mathcal{D} = f\,D_x + g\,D_y, \tag{35}$$

and Eqs. (34) for the coefficients can be written as

$$\begin{aligned} X(f) &= fD_x(\xi(x)) + gD_y(\xi(x)) \equiv -\xi'(x)f, \qquad X(g) = 0,\\ Y(g) &= fD_x(\eta(y)) + gD_y(\eta(y)) \equiv -\eta'(y)g, \qquad Y(f) = 0. \end{aligned} \tag{36}$$

Here f and g are unknown function x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, $\ldots$. It is implied that the operators X and Y are extended to all derivatives of h and k being considered.

We begin with the case where $f = f(x, y, h, k)$ and $g = g(x, y, h, k)$. Then, Eqs. (36) give the following system of equations for f:

$$\xi\,\frac{\partial f}{\partial x} - \xi'\Big[h\,\frac{\partial f}{\partial h} + k\,\frac{\partial f}{\partial k}\Big] = \xi'(x)f, \qquad \eta\,\frac{\partial f}{\partial y} - \eta'\Big[h\,\frac{\partial f}{\partial h} + k\,\frac{\partial f}{\partial k}\Big] = 0.$$

As in Sec. 2, using the fact that ξ, ξ', η, and η' are arbitrary functions, we arrive at the following four equations:

$$\frac{\partial f}{\partial x} = 0, \qquad h\,\frac{\partial f}{\partial h} + k\,\frac{\partial f}{\partial k} = -f, \qquad \frac{\partial f}{\partial y} = 0, \qquad h\,\frac{\partial f}{\partial h} + k\,\frac{\partial f}{\partial k} = 0,$$

whence it immediately follows that $f = 0$. Similarly, Eqs. (36), written for $g = g(x, y, h, k)$, give $g = 0$. This means that there are no invariant differentiations of (35) with the coefficients $f = f(x, y, h, k)$ and $g = g(x, y, h, k)$.

Therefore, we continue the search by setting

$$f = f(x, y, h, k, h_x, h_y, k_x, k_y), \qquad g = g(x, y, h, k, h_x, h_y, k_x, k_y).$$

The extended operators X and Y leads to the following operators [compare (11) and (13)]

$$X_\xi = \frac{\partial}{\partial x}, \qquad X_{\xi''} = h\,\frac{\partial}{\partial h_x} + k\,\frac{\partial}{\partial k_x},$$
$$X_{\xi'} = h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k} + 2h_x\,\frac{\partial}{\partial h_x} + h_y\,\frac{\partial}{\partial h_y} + 2k_x\,\frac{\partial}{\partial k_x} + k_y\,\frac{\partial}{\partial k_y} \tag{37}$$

and, hence, to the operators

$$Y_\eta = \frac{\partial}{\partial y}, \qquad Y_{\eta''} = h\,\frac{\partial}{\partial h_y} + k\,\frac{\partial}{\partial k_y},$$
$$Y_{\eta'} = h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k} + h_x\,\frac{\partial}{\partial h_x} + 2h_y\,\frac{\partial}{\partial h_y} + k_x\,\frac{\partial}{\partial k_x} + 2k_y\,\frac{\partial}{\partial k_y}. \tag{38}$$

The existence of the operators X_ξ and X_η leads to the fact that f and g do not depend on x and y. Next, Eqs. (36) split into the equations

$$X_{\xi'}(f) = -f, \qquad X_{\xi''}(f) = 0, \qquad Y_{\eta'}(f) = 0, \qquad Y_{\eta''}(f) = 0 \tag{39}$$

for the function $f(h, k, h_x, h_y, k_x, k_y)$ and the equations

$$X_{\xi'}(g) = 0, \qquad X_{\xi''}(g) = 0, \qquad Y_{\eta'}(g) = -g, \qquad Y_{\eta''}(g) = 0 \tag{40}$$

for the function $g(h, k, h_x, h_y, k_x, k_y)$. Of them, the pair of equations $X_{\xi''}(f) = 0$ and $Y_{\eta''}(f) = 0$ for f and the pair of equations $X_{\xi''}(g) = 0$ and $Y_{\eta''}(g) = 0$ for g show that f and g depend only on the following four variables (compare Sec. 2):

$$h, \quad k, \quad \lambda = k_x - p h_x = h p_x, \quad \mu = k_y - p h_y = h p_y.$$

We now rewrite the operators $X_{\xi'}$ and $Y_{\eta'}$ in the variables h, λ, μ, and $p = k/h$:

$$X_{\xi'} = h\,\frac{\partial}{\partial h} + 2\lambda\,\frac{\partial}{\partial \lambda} + \mu\,\frac{\partial}{\partial \mu}, \qquad Y_{\eta'} = h\,\frac{\partial}{\partial h} + \lambda\,\frac{\partial}{\partial \lambda} + 2\mu\,\frac{\partial}{\partial \mu} \tag{41}$$

and integrate the equations

$$X_{\xi'}(f) = -f, \qquad Y_{\eta'}(f) = 0$$

for the function $f(h, p, \lambda, \mu)$ and similar equations

$$X_{\xi'}(g) = 0, \qquad Y_{\eta'}(g) = -g$$

for $g(h, p, \lambda, \mu)$. As a result, we obtain

$$f = \frac{h}{\lambda}\,F(p, I), \qquad g = \frac{h}{\mu}\,G(p, I), \tag{42}$$

where $\lambda = h p_x$, $\mu = h p_y$, and p and I are invariants of (15) and (30):

$$p = \frac{k}{h}, \qquad I = \frac{\lambda\mu}{h^3} = \frac{p_x p_y}{h}.$$

Substitution of expressions (42) into (35) leads to the invariant-differentiation operator

$$\mathcal{D} = F(p, I)\,\frac{1}{p_x}\,D_x + G(p, I)\,\frac{1}{p_y}\,D_y \tag{43}$$

with arbitrary functions $F(p, I)$ and $G(p, I)$.

Remark 1. The most general invariant differentiation has the form of (43) with $F(p, I)$ and $G(p, I)$ replaced by arbitrary functions of higher-order invariants [or example, by $F(p, I, q, \tilde{q}, N, H)$ and $G(p, I, q, \tilde{q}, N, H)$] if the corresponding invariants are found. This, however, is not needed; it only suffices to set $F = \text{const}$ and $G = \text{const}$.

Setting $F = 1$ and $G = 0$ and then $F = 0$ and $G = 1$ in (43), we obtain the following simple invariant differentiations in the x and y directions:

$$\mathcal{D}_x = \frac{1}{p_x} D_x, \qquad \mathcal{D}_y = \frac{1}{p_y} D_y. \tag{44}$$

It is now possible to construct higher-order invariants using the invariant differentiations (44) and to prove the following statement.

Theorem 1. *The basis of invariants of arbitrary order for Eq.* (1) *consists of the invariants*

$$p = \frac{k}{h}, \qquad I = \frac{p_x\, p_y}{h}, \qquad q = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x\, \partial y}, \qquad \tilde{q} = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x\, \partial y} \tag{45}$$

or the alternative basis invariants

$$p = \frac{k}{h}, \qquad I = \frac{p_x p_y}{h}, \qquad N = \frac{1}{p_x}\,\frac{\partial}{\partial x}\,\ln\left|\frac{p_x}{h}\right|, \qquad q = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x\, \partial y}. \tag{46}$$

Proof. Simple calculations lead to the relations

$$\mathcal{D}_x(p) = 1, \qquad \mathcal{D}_x(I) = (N + 1/p)I + p(p\tilde{q} - q),$$

$$\mathcal{D}_y(p) = 1, \qquad \mathcal{D}_y(I) = (H + 1/p)I + p(p\tilde{q} - q).$$

They show that invariants (46) can be obtained from (45) using invariant differentiations and vice versa. Hence, as the basis of all second-order invariants (31), (32), it is possible to choose either (45) or (46). Next, using Eqs. (24), one can show that the invariant differentiations $\mathcal{D}_x$ and $\mathcal{D}_y$ of the basis invariants (45) or (46) yield six independent invariants which depend on the third-order partial derivatives of h and k. At the same time, the consideration of the third-order invariants involves accounting for eight third-order derivatives of h and k. However, the number of invariance conditions taking into account the fourth derivatives $\xi^{(iv)}(x)$ and $\eta^{(iv)}(y)$ increases by two equations, so that, eventually, only six additional invariants remain — just as many invariants as obtained by invariant differentiations. The same reasoning in the case of derivatives of any order completes the proof of the theorem.

REFERENCES

1. P. S. Laplace, "Recherches sur le calcul int égral aux différences partielles," in: *Mémoires de l'Acad émie royale des Sciences de Paris* (1773/77), pp. 341–402; reprinted from P. S. Laplace, *Oeuvres Complètes*, Vol. 9, Gauthier-Villars, Paris (1893).
2. N. H. Ibragimov, "Infinitesimal method in the theory of invariants of algebraic and differential equations," *Notices South African Math. Soc.*, **29**, 61–70 (1997).
3. E. Laguerre, "Sur quelques invariants des équations différentielles lin éaires," *Comp. Rend. Acad. Sci. Paris*, **88**, 224–227 (1879).
4. L. V. Ovsyannikov, "Group properties of the Chaplygin equation," *J. Appl. Mech. Tech. Phys.*, **3**, 126–145 (1960).
5. N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons, Chichester (1999).
6. N. H. Ibragimov, "Laplace type invariants for parabolic equations," *Nonlinear Dyn.*, **28**, No. 2, 125–133 (2002).
7. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).